Improper Integrals

Name: Block: Seat:

Improper Integrals are integrals where:

- One or both limits of integration are $\pm\infty$
  (sometimes called "Type I" or "horizontal" improper integrals)
- The integrand is undefined at one or both of the limits of integration or in between the limits of integration
  (the integrand has a vertical asymptote, sometimes called "Type II" or "vertical" improper integrals)

Improper integrals are said to be

- **convergent** if the limit is finite and that limit is the value of the improper integral.
- **divergent** if the limit does not exist.

Examples

1. $\displaystyle\int_0^{\infty} e^{-x}\ dx = \lim_{b\to\infty}\int_0^{b} e^{-x}\ dx = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = \lim_{b\to\infty}\frac{-1}{e^b} + e^0 = 0 + 1 = 1$

2. $\displaystyle\int_{-\infty}^{\infty}\frac{1}{1+x^2}\ dx = \int_{-\infty}^{0}\frac{1}{1+x^2}\ dx + \int_0^{\infty}\frac{1}{1+x^2}\ dx$

   First we consider $\displaystyle\int_{-\infty}^{0}\frac{1}{1+x^2}\ dx = \lim_{b\to-\infty}\left[\arctan x\right]_b^0 = \arctan 0 - \lim_{b\to-\infty}\arctan b = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$

   Similarly $\displaystyle\int_0^{\infty}\frac{1}{1+x^2}\ dx = \lim_{b\to\infty}\left[\arctan x\right]_0^b = \lim_{b\to\infty}\arctan b - \arctan 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

   Putting it together, $\displaystyle\int_{-\infty}^{\infty}\frac{1}{1+x^2}\ dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

p-Test for Integrals Thm 8.7 (page 578):

If $p > 1$, then $\int_1^\infty \frac{1}{x^p}\,dx$ converges to $\frac{1}{p-1}$

If $p \leq 1$, then $\int_1^\infty \frac{1}{x^p}\,dx$ diverges

Examples

1. $\int_1^\infty \frac{1}{x^{3/2}}\,dx$

   Following usual methods:

   $$\lim_{b\to\infty}\int_1^b \frac{1}{x^{3/2}}\,dx = \lim_{b\to\infty}\int_1^b x^{-3/2}\,dx = \lim_{b\to\infty}\left[\frac{1}{-\frac{1}{2}}x^{-1/2}\right]_1^b = \lim_{b\to\infty} -\left[\frac{2}{x^{1/2}}\right]_1^b = 0-(-2) = 2$$

   But following the theorem:

   $p = \frac{3}{2} > 1$, hence $\frac{1}{p-1} = \frac{1}{\frac{3}{2}-1} = \frac{1}{\frac{1}{2}} = 2$

2. $\int_1^\infty \frac{1}{x^{1/2}}\,dx$

   Following usual methods:

   $$\lim_{b\to\infty}\int_1^b \frac{1}{x^{1/2}}\,dx = \lim_{b\to\infty}\left[2\sqrt{x}\right]_1^b = \lim_{b\to\infty} 2\sqrt{b}-2 = \infty$$

   But following the theorem:

   $p = \frac{1}{2} \leq 1$, hence it diverges

Exercises

1. $\displaystyle\int_1^{\infty} \frac{1}{x}\, dx$ (try it both ways!)

2. $\displaystyle\int_1^{\infty} \frac{1}{x^2}\, dx$

3. $\displaystyle\int_0^{\infty} \frac{1}{x^2+9}\, dx$

   Hint: $\displaystyle\frac{1}{x^2+9} = \frac{\frac{1}{9}}{\frac{x^2}{9}+1}$ reminds me of arctan (see p. 520)

4. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$
Hint: Use the limit as $b \to 0^+$

5. $\displaystyle\int_{-1}^1 \frac{1}{x^{2/3}}\,dx$

6. $\displaystyle\int_{-2}^1 \frac{1}{x^2}\,dx$

7. $\int_0^1 x \ln x \; dx$

8. $\int_0^\infty \cos x \; dx$

Answers: 1) diverges, 2) 1 3) $\frac{\pi}{6}$ 4) 2 5) 6 6) diverges 7) $-\frac{1}{4}$ 8) diverges